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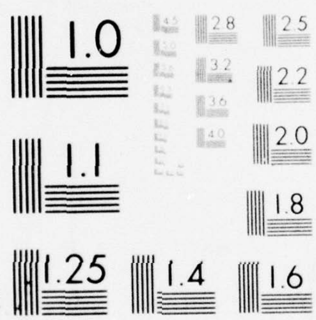

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The Inverted Complex Wishart Distribution And Its Application To Spectral Estimation

by

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Abstract

The inverted complex Wishart distribution is studied and its use for the construction of spectral estimates is illustrated. The density, some marginals of the distribution, and the first- and second-order moments are given. For a vector-valued time series, estimation of the spectral density at a collection of frequencies and estimation of the increments of the spectral distribution function in each of a set of frequency bands are considered. A formal procedure applies Bayes theorem, where the complex Wishart is used to represent the distribution of an average of adjacent periodogram values. A conjugate prior distribution for each parameter vector is a product of inverted complex Wishart distributions.

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Key words and phrases: Complex Wishart distribution, inverted complex Wishart distribution, multiple time series, spectral density, periodogram, prior distribution, posterior distribution.

1. Introduction

In multiple time series analysis complex multivariate distributions are commonly used to describe estimates of frequency domain parameters. A review of complex multivariate distributions and their application in time series has recently been given by Krishnaiah [8]. The complex Wishart distribution, in particular, was introduced and used by Goodman [4,5] to approximate the distribution of an estimate of the spectral density matrix for a vector-valued stationary Gaussian process. In this paper the inverted complex Wishart distribution is studied and its use for the construction of spectral estimates is illustrated.

Methods of spectral estimation typically involve periodogram smoothing. The amount and type of smoothing one performs depend to a considerable extent upon prior knowledge of the spectral density to be estimated. A method of incorporating prior information about the shape and smoothness of a spectral density into the formation of a spectral estimate has been given by Shaman [9] for a univariate time series. Two types of finite-dimensional parameters are considered, the spectral density ordinates at a specified collection of frequencies and the amount of power in each of a set of frequency bands. The method is conditional upon the asymptotic distribution of periodogram averages. A formal procedure applies Bayes theorem, with a conjugate prior distribution being a product of inverted gamma distributions. The mean of the posterior distribution involves simple linear adjustments of the periodogram averages, with coefficients depending upon prior distribution parameters. Although the method is not genuinely Bayesian, it does permit one to incorporate prior information about the height and shape of the spectral density into the construction of an estimate in a formal manner.

The spectral density estimation methodology just discussed is extended to a vector time series model in the present paper. The asymptotic distribution of a set of nonoverlapping periodogram averages is a product of complex Wishart distributions. A conjugate prior distribution is a product of inverted complex Wishart distributions.

In Section 2 the density of the inverted complex Wishart distribution will be derived, as well as some marginals of the distribution and its first- and second-order moments. Details of the proposed use of the inverted complex Wishart distribution in spectral estimation are given in Section 3.

2. The Inverted Complex Wishart Distribution

Let $X_1,\cdots,X_n$ be independent $r \times 1$ vectors, each complex normal with mean 0 and covariance matrix Σ (see Wooding [12] and Goodman [5]). Then the $r \times r$ matrix

$$W = \sum_{j=1}^{n} X_j X_j^* ,$$

where the asterisk designates conjugate transpose, has a complex Wishart distribution with n degrees of freedom and covariance matrix Σ, denoted $W_C(r,n,\Sigma)$. The density is (Goodman [5])

$$\frac{1}{\tilde{\Gamma}_r(n)|\Sigma|^n} |W|^{n-r} \operatorname{etr}(-\Sigma^{-1}W), \quad n \geq r, \; W \geq 0, \tag{1}$$

where

$$\tilde{\Gamma}_r(n) = \pi^{\frac{1}{2}r(r-1)} \prod_{j=1}^{r} \Gamma(n-j+1)$$

is the complex multivariate gamma function.

We wish to determine the density of $V = W^{-1}$, an inverted complex Wishart variate. The volume element associated with a Hermitian matrix $C = (c_{jkR} + ic_{jkI})$ is $\Pi_j dc_{jjR} \Pi_{k>j} dc_{jkI}$.

Lemma. Let X be an $r \times r$ Hermitian nonsingular matrix. Then the Jacobian of the transformation $Y = X^{-1}$ is $|Y|^{-2r}$.

Proof. Let A be an $r \times r$ complex nonsingular matrix. The Jacobian of the transformation $Y = AXA^*$ is $|AA^*|^{-r}$, by (2.8) of Khatri [7]. The desired result follows from consideration of $dY = dX^{-1} = -X^{-1}dX\,X^{-1} = -X^{-1}dX(X^*)^{-1}$.

According to the lemma the density of V is

$$\frac{|\Psi|^n}{\tilde{\Gamma}_r(n)} \frac{\operatorname{etr}(-V^{-1}\Psi)}{|V|^{n+r}}, \quad n \geq r, \; V > 0, \tag{2}$$

where $\Psi = \Sigma^{-1}$. Denote the distribution of V by $W_C^{-1}(r,n,\Psi)$.

The marginal distributions of certain sets of elements of V are also inverted complex Wishart. Let

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

where V_{11} is $q \times q$, and partition Ψ similarly. Consider the distribution of V_{11}. The method of derivation is that given by Tiao and Zellner [10] for marginals of the inverted Wishart distribution, modified for the present complex case. The Jacobian of the transformation from V_{12}, V_{22} to G, H,

$$G = V_{11}^{-1} V_{12}, \quad H = V_{22} - V_{21} V_{11}^{-1} V_{12}$$

is $|V_{11}|^{2(r-q)}$, from (2.3) of Khatri [7]. Then the marginal density of V_{11} is seen to be

$$\frac{|\Psi_{11}|^{n-r+q}}{\tilde{\Gamma}_q(n-r+q)} \frac{\operatorname{etr}(-V_{11}^{-1}\Psi_{11})}{|V_{11}|^{n-r+2q}}, \quad 1 \le q \le r \le n, \quad V_{11} > 0.$$

That is, V_{11} is $W_C^{-1}(q, n-r+q, \Psi_{11})$. When $q=1$, V_{11} has the inverted gamma distribution with density

$$\frac{\psi_{11}^{n-r+1} \exp(-\psi_{11}/V_{11})}{\Gamma(n-r+1)\, V_{11}^{n-r+2}}, \quad n \ge r, \ V_{11} > 0.$$

Next we consider first- and second-order moments of V. The derivation follows that given by Kaufman [6] for the real inverted Wishart distribution. The method is to write the complex Wishart matrix W as TT^*, where T is lower triangular. The joint distribution of the elements of T is easily obtained when $\Sigma = I$ and can be used to derive moments of T^{-1} in the general case.

If the $r \times r$ Hermitian matrix W is written as TT^*, where T is lower triangular, the transformation from W to T has Jacobian

$$2^r \prod_{j=1}^{r} t_{jj}^{2r-2j+1}$$

(see Goodman [5], p.165). Then by (1) the density of T is

$$\frac{2^r}{\tilde{\Gamma}_r(n)|\Sigma|^n} \prod_{j=1}^{r} t_{jj}^{2n-2j+1} \operatorname{etr}(-\Sigma^{-1}TT^*). \tag{3}$$

When $\Sigma = I$, (3) becomes

$$\prod_{j=1}^{r} \frac{2}{\Gamma(n-j+1)} \exp(-t_{jj}^2)\, t_{jj}^{2n-2j+1} \prod_{j=2}^{r} \prod_{k=1}^{j-1} \frac{1}{\pi}\exp(-|t_{jk}|^2), \tag{4}$$

which is the density of $\frac{1}{2}r(r+1)$ independent random variables. Specifically, $2t_{jj}^2$ is $\chi^2_{2n-2j+2}$, $j=1,\cdots,r$, and $t_{jk} = t_{jkR} + it_{jkI}$ has a univariate complex normal distribution, $j > k$. The pair (t_{jkR}, t_{jkI}) is bivariate normal with mean 0 and covariance matrix $\frac{1}{2}I_2$.

The inverted complex Wishart matrix $V = W^{-1}$ is S^*S, where $S = (s_{jk}) = T^{-1}$ is lower triangular. To clarify some later discussion we denote W^{-1} by Y when $\Sigma = I$, that is, when T has density (4).

In terms of elements of T,

$$s_{jj} = \frac{1}{t_{jj}}, \qquad j=1,\cdots,r,$$

$$s_{jk} = \frac{1}{t_{kk}}\left(-u_{jk} + \sum_{\ell_1=k+1}^{j-1} u_{j\ell_1}u_{\ell_1 k} - \sum_{\ell_2=k+1}^{j-2}\sum_{\ell_1=\ell_2+1}^{j-1} u_{j\ell_1}u_{\ell_1\ell_2}u_{\ell_2 k}\right.$$

$$\left. +\cdots+(-1)^{j-k}u_{j,j-1}u_{j-1,j-2}\cdots u_{k+1,k}\right), \quad j>k, \tag{5}$$

where

$$u_{jk} = t_{jk}/t_{jj}, \qquad j>k. \tag{6}$$

When $\Sigma = I$ the distribution of $(2n-2j+2)^{\frac{1}{2}}u_{jk}$ is complex t with $2n-2j+2$ degrees of freedom. The density of this complex t variate with f degrees of freedom is

$$g(u) = \frac{1}{2\pi\left(1+\frac{|u|^2}{f}\right)^{\frac{1}{2}f+1}}.$$

Some moments of the variables t_{jj}, t_{jk}, and u_{jk} when $\Sigma = I$ are required. First note

$$E(t_{jj}^{-2}) = \frac{1}{n-j}, \quad E(t_{jj}^{-4}) = \frac{1}{(n-j)(n-j-1)}, \tag{7}$$

$$j=1,\cdots r, \quad n>r+1.$$

The complex normal variables t_{jk} satisfy

$$E(|t_{jk}|^2)=1, \quad E(|t_{jk}|^4)=2, \quad j>k, \tag{8}$$

and these are the only nonzero moments up to order four. That is, all odd moments of t_{jk} are 0, and moreover $E(t_{jk}^2) = 0$, $E(t_{jk}^4) = E(t_{jk}^3 \bar{t}_{jk}) = 0$. For the u_{jk} variables, (7) and (8) imply

$$E(|u_{jk}|^2) = \frac{1}{n-j}, \quad E(|u_{jk}|^4) = \frac{2}{(n-j)(n-j-1)}, \qquad j > k, \quad n > r+1, \tag{9}$$

and these are the only nonzero moments up to order four, as with the variables t_{jk}.

The elements of Y are

$$y_{jk} = \sum_{g=\max(j,k)}^{r} \bar{s}_{gj} s_{gk}, \qquad j \geq k. \tag{10}$$

By (4) - (6) and the discussion following (8), we see that $E(\bar{s}_{gj} s_{gk}) = 0$, $j \neq k$. Therefore

$$E(y_{jk}) = 0, \qquad j \neq k. \tag{11}$$

By (4) - (10),

$$E(|s_{gj}|^2) = \frac{1}{n-j}\left(\frac{1}{n-g} + \sum_{\ell_1=j+1}^{g-1} \frac{1}{n-g}\,\frac{1}{n-\ell_1} + \sum_{\ell_2=j+1}^{g-2} \sum_{\ell_1=\ell_2+1}^{g-1} \frac{1}{n-g}\,\frac{1}{n-\ell_1}\,\frac{1}{n-\ell_2} + \cdots + \frac{1}{n-g}\,\frac{1}{n-g+1}\cdots\frac{1}{n-j-1}\right). \tag{12}$$

It follows from (5) and (6) with $t_{jj} = n-j$, $j=1,\cdots,r$, $t_{jk} = -1$, $j>k$, and (12) that the $r \times r$ matrix with elements $E(|s_{gj}|^2)$ is the inverse of

$$\begin{bmatrix} n-1 & 0 & 0 & \cdots & 0 \\ -1 & n-2 & 0 & \cdots & 0 \\ -1 & -1 & n-3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & n-r \end{bmatrix},$$

which is

$$\begin{bmatrix} \frac{1}{n-1} & 0 & 0 & \cdots & 0 \\ \frac{1}{(n-1)(n-2)} & \frac{1}{n-2} & 0 & \cdots & 0 \\ \frac{1}{(n-2)(n-3)} & \frac{1}{(n-2)(n-3)} & \frac{1}{n-3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{(n-r+1)(n-r)} & \frac{1}{(n-r+1)(n-r)} & \frac{1}{(n-r+1)(n-r)} & \cdots & \frac{1}{n-r} \end{bmatrix}.$$

Therefore

$$\begin{aligned} E(y_{jj}) &= \sum_{g=j}^{r} E(|s_{gj}|^2) \\ &= \frac{1}{n-j} + \sum_{i=0}^{r-j-1} \frac{1}{(n-j-i)(n-j-i-1)} \qquad (13) \\ &= \frac{1}{n-r}, \qquad j=1,\cdots,r. \end{aligned}$$

Evaluation of second-order moments of the elements of Y is much more difficult and tedious. From the above discussion it follows that

$$\mathrm{Cov}(y_{jk}, y_{\ell m}) = 0 \quad \text{unless} \quad j=k, \ell=m, \quad \text{or} \quad j=m, k=\ell. \tag{14}$$

Lengthy direct calculations from (4) - (11) and (13) yield

$$\mathrm{Var}(y_{jj}) = \frac{1}{(n-r)^2(n-r-1)}, \quad j=1,\cdots,r, \quad n>r+1, \tag{15}$$

$$\mathrm{Cov}(y_{jj}, y_{kk}) = \frac{1}{(n-r+1)(n-r)^2(n-r-1)}, \quad j \neq k, \quad n>r+1, \tag{16}$$

and

$$\mathrm{Cov}(y_{jk}, y_{kj}) = \frac{1}{(n-r+1)(n-r)(n-r-1)}, \quad j \neq k, \quad n>r+1. \tag{17}$$

Now let Σ be an arbitrary Hermitian positive definite matrix. Write $\Sigma = AA^*$, where A is lower triangular. If W is $W_C(r,n,I)$, then AWA^* is $W_C(r,n,\Sigma)$. The desired inverse is $V = B^*YB$, where $B = A^{-1}$ and Y is $W_C^{-1}(r,n,I)$. Thus

$$v_{jk} = \sum_{\alpha=j}^{r} \sum_{\beta=k}^{r} \bar{b}_{\alpha j} y_{\alpha\beta} b_{\beta k}, \qquad j,k=1,\cdots,r. \tag{18}$$

From (11), (13) - (18), and some further manipulations we obtain the desired moments of V, denoting $\Sigma^{-1} = (\sigma^{jk})$,

$$E(V) = \frac{1}{n-r}\Sigma^{-1}, \qquad n>r, \tag{19}$$

$$\mathrm{Cov}(v_{jk}, v_{\ell m}) = \frac{\sigma^{jm}\sigma^{\ell k} + \frac{1}{n-r}\sigma^{jk}\sigma^{\ell m}}{(n-r+1)(n-r)(n-r-1)}, \qquad n>r+1. \tag{20}$$

3. Application of the Inverted Complex Wishart Distribution to Spectral Estimation

The methodology described below is based upon distributional approximations and we make assumptions which allow these to hold. The conditions in Brillinger, [1] and [2], Chapters 5 and 7, in particular, are used.

Let $X(t)$ $(t=0,\pm 1,\cdots)$ be a vector-valued strictly stationary stochastic process for which all moments exist. Denote the components of $X(t)$ by $X_j(t)$ $(j=1,\cdots,r)$, the mean by $E\{X(t)\}=m$, and the spectral density by $f(\lambda)$ $(-\pi\leq\lambda\leq\pi)$. The cumulant functions of the process are

$$C_{j_1\cdots j_k}(t_1,\cdots,t_{k-1})=\operatorname{cum}\{X_{j_1}(t_1+t),\cdots,X_{j_{k-1}}(t_{k-1}+t),X_{j_k}(t)\}$$

$$(j_i=1,\cdots,r,\ i=1,\cdots,k,\ t_1+t,\cdots,t_{k-1}+t,\ t=0,\pm 1,\cdots,k=2,3,\cdots).$$

Assume for $j_i=1,\cdots,r(i=1,\cdots,k)$ that

$$\sum_{t_1,\cdots,t_{k-1}=-\infty}^{\infty}|C_{j_1,\cdots,j_k}(t_1,\cdots,t_{k-1})|<\infty\quad(k=2,3,\cdots). \tag{21}$$

This ensures the existence and uniform continuity of cumulant spectra of all orders.

Assume a time series $X(t)$ $(t=0,1,\cdots,T-1)$ is available. The periodogram is

$$I(\lambda)=\frac{1}{2\pi T}Z(\lambda)Z(\lambda)^*\qquad(-\pi\leq\lambda\leq\pi), \tag{22}$$

where $Z(\lambda) = \sum_{t=0}^{T-1} e^{i\lambda t} x(t)$ $(-\pi \leq \lambda \leq \pi)$. Let $p = [\frac{1}{2}(T-1)]$. Then $I(2\pi j/T)$ are asymptotically independent variables distributed as $W_C\{r, 1, f(2\pi j/T)\}$ $(j=1, \cdots, p)$ (see Brillinger [2], Theorem 7.2.4). Also $I(\pi)$ is asymptotically an $r \times r$ Wishart variable with one degree of freedom and covariance matrix $f(\pi)$, and is independent of the other variables. If $m \neq 0$, $I(0)$ is approximately an $r \times r$ noncentral Wishart variable with one degree of freedom.

Restrict attention to frequencies $0 \leq \lambda \leq \pi$ and let $j(T)$ be a sequence of integers such that $2\pi j(T)/T$ is near $\lambda (\neq 0, \pi)$ and converges to λ as $T \to \infty$. Then

$$z = \frac{1}{2n+1} \sum_{h=-n}^{n} I[2\pi\{j(T)+h\}/T] \tag{23}$$

is an estimate of $f(\lambda)$ and is asymptotically distributed as $(2n+1)^{-1} W_C\{r, 2n+1, f(\lambda)\}$. If $\lambda = 0$ (23) is replaced by

$$z = \frac{1}{n} \sum_{h=1}^{n} I(2\pi h/T), \tag{24}$$

and if $\lambda = \pi$,

$$\begin{aligned} z &= \frac{1}{n} \sum_{h=1}^{n} I(\pi - 2\pi h/T) \qquad (T \text{ even}) \\ &= \frac{1}{n} \sum_{h=1}^{n} I(\pi - \pi/T - 2\pi h/T) \qquad (T \text{ odd}) \end{aligned} \tag{25}$$

is used. The complex Wishart was established as a limiting distribution for (22) and for (23) - (25) with fixed n by Brillinger [1] for the case $m = 0$. For $\lambda \neq 0$, the asymptotic distributions of $I(\lambda)$ and z are the same whether or not $m = 0$ and in [2] Brillinger treats an arbitrary m. Wahba [11] and Gleser and Pagano [3] allow $n \to \infty$ under the assumption $X(t)$ is Gaussian. Under appropriate conditions, M nonoverlapping sums of the form (23) are asymptotically independent complex Wishart matrices as $n, M, T \to \infty$. The covariance matrices of the asymptotic complex Wishart distributions are

$$\frac{1}{2n+1} \sum_{h=-n}^{n} f[2\pi\{j(T)+h\}/T],$$

where $2\pi j(T)/T$ converges to some λ as $T \to \infty$.

Consider estimation of the spectral density at a fixed, preassigned set of frequencies, $0 \leq \lambda_1 < \cdots < \lambda_M \leq \pi$. The choice of M and the frequencies may involve use of prior information. For example, if the spectral density is considered a priori to be approximately constant in certain bands, the frequencies may be interior points of the bands.

To avoid anomalous cases assume $\lambda_1 > 0$, $\lambda_M < \pi$. Let z_ℓ denote the right-hand side of (23) for $2n+1 \geq r$ and $2\pi j(T)/T$ near $\lambda_\ell (\ell=1,\cdots,M)$. Define $z = (z_1,\cdots,z_M)$ and $f = \{f(\lambda_1),\cdots,f(\lambda_M)\}$. We use the asymptotic distribution of z described above. Then z has density

$$h(z|f) = \frac{(2n+1)^{(2n+1)rM}}{\{\tilde{\Gamma}_r(2n+1)\}^M} \prod_{\ell=1}^{M} \frac{|z_\ell|^{2n+1-r}}{|f(\lambda_\ell)|^{2n+1}} \operatorname{etr}\{-f(\lambda_\ell)^{-1}(2n+1)z_\ell\} \tag{26}$$

$$(2n+1 \geq r).$$

Then a conjugate prior density is a product of inverted complex Wisharts,

$$h(f) = \prod_{\ell=1}^{M} \frac{|\beta_\ell|^{\alpha_\ell} \operatorname{etr}\{-f(\lambda_\ell)^{-1}\beta_\ell\}}{\tilde{\Gamma}_r(\alpha_\ell)|f(\lambda_\ell)|^{\alpha_\ell+r}} \quad (\alpha_\ell \geq r,\ \ell=1,\cdots,M). \tag{27}$$

The posterior density from (26) and (27) is

$$h(f|z) = \prod_{\ell=1}^{M} \frac{|(2n+1)z_\ell+\beta_\ell|^{2n+1+\alpha_\ell} \operatorname{etr}[-f(\lambda_\ell)^{-1}\{(2n+1)z_\ell+\beta_\ell\}]}{\tilde{\Gamma}_r(2n+1+\alpha_\ell)|f(\lambda_\ell)|^{2n+r+1+\alpha_\ell}}, \tag{28}$$

a product of inverted complex Wishart densities. The mean of the posterior occurs at $f(\lambda_\ell) = \{(2n+1)z_\ell + \beta_\ell\}/(2n+1+\alpha_\ell-r)$ $(\ell=1,\cdots,M)$.

Knowledge of the spectral density ordinate at each of a number of specified frequencies can convey an accurate picture of the shape of the curve. However, more basic interest may concern the increments of the spectral distribution function in certain frequency bands. Consider a partition $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_M < \lambda_{M+1} = \pi$ and let $F(\lambda)$ denote the spectral distribution function. Consider the parameter $p = (p_1,\cdots,p_{M+1})$, where $p_\ell = F(\lambda_\ell) - F(\lambda_{\ell-1})$ $(\ell=1,\cdots,M+1)$. The partition is fixed for all sample sizes. Let $k_\ell(T)\{\ell=0,\cdots,M+1,\ k_0(T)=1,\ k_{m+1}(T)=[\frac{1}{2}(T-1)]\}$ be integers such that the frequencies $2\pi j/T$ are in the interval $(\lambda_{\ell-1},\lambda_\ell)$ for

$k_{\ell-1}(T) \leq j \leq k_\ell(T)-1$ and define $m_\ell(T) = k_\ell(T) - k_{\ell-1}(T)$. Then under the assumption (21) and the conditions in Wahba [11] or Gleser and Pagano [3] the sums

$$y_\ell = \sum_{j=k_{\ell-1}(T)}^{k_\ell(T)-1} I(2\pi j/T) \qquad (\ell=1,\ldots,M+1)$$

are approximately distributed as independent $r \times r$ complex Wishart variables with $m_\ell(T)$ degrees of freedom and covariance matrices

$$\frac{1}{m_\ell(T)} \sum_{j=k_{\ell-1}(T)}^{k_\ell(T)-1} f(2\pi j/T) \qquad (\ell=1,\ldots,M+1).$$

We further approximate the distribution of y_ℓ as that of $W_C\{r, m_\ell(T), Tp_\ell/(2\pi m_\ell(T))\}$ $(\ell=1,\ldots,M+1)$. Details of the transition to a posterior distribution for the parameter p are similar to those given at (26) - (28). One can restrict attention to a set of frequency bands whose union forms a subset of $[0,\pi]$.

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The inverted complex Wishart distribution is studied and its use for the construction of spectral estimates is illustrated. The density, some marginals of the distribution, and the first- and second-order moments are given. For a vector-values time series, estimation of the spectral density at a collection of frequencies and estimation of the increments of

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the spectral distribution function in each of a set of frequency bands are considered. A formal procedure applies Bayes theorem, where the complex Wishart is used to represent the distribution of an average of adjacent periodogram values. A conjugate prior distribution for each parameter vector is a product of inverted complex Wishart distributions.